

Helix, shadow boundary and minimal submanifolds

Abstract

Inspired by Blaschke's work about analytic convex surfaces, we study *shadow boundaries* of Riemannian submanifolds M , which are defined by a parallel vector field along M . Since a shadow boundary is just a closed subset of M , first, we will give a condition that guarantee its smoothness. It depends on the second fundamental form of the submanifold. It is natural to search for what kind of properties might have such submanifolds of M ? Could they be totally geodesic or minimal? Answers to these and related questions are given in this work.

Key words: minimal and helix submanifolds, smooth shadow boundaries, parallel vector field, holonomy group.

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1 Introduction

In their book [12], Nomizu and Sasaki defined the shadow boundary of a Euclidean surface S with respect to a fixed direction v as the set of points $p \in S$ such that: the line through p in the direction of v is tangent to S (see also [4]). Blaschke characterized convex analytic surfaces with planar shadow boundaries. We can see an extension of his result in [12], page 61.

Recently M. Ghomi in [6], solved the shadow problem formulated by H. Wente. The shadow of a Euclidean orientable surface S , with unit normal vector field $n : S \rightarrow S^2$, consist of those points $p \in S$ where the inner product between $n(p)$ and v is positive. His result says that a closed surface with simply connected shadows should be convex (see also [9]). He used *horizon* term in instead of shadow boundary.

In a previous work [15], we studied shadow boundaries of any Euclidean submanifold. Now, we extend the concept of shadow boundary in the following way. Let N be a Riemannian manifold and let $M \subset N$ be a submanifold. Let us assume that $Y : M \rightarrow TN$ is an invariant vector field under parallel transport in N along curves contained in M , in Theorem 2.1 we give a condition in terms of the holonomy group of N for the existence of such vector field Y . For example Y could be the restriction

to M of a global parallel vector field on N . The *shadow boundary* of M with respect to Y is the following subset of M .

$$S\partial(M, Y) = \{x \in M \mid Y(x) \in T_x M\},$$

i.e. the points where the vector field is tangent to the submanifold. This is a closed subset of M . In Theorem 4.2, there is a condition over the second fundamental form of $M \subset N$ that makes $S\partial(M, Y)$ a submanifold of M . A consequence is that $S\partial(M, Y)$ could be a finite set of points, if M is compact with $\dim N = 2 \dim M$.

When we were looking for properties of these type of submanifolds of M , we observed the necessity to introduce Helix submanifolds of a Riemannian manifold: Submanifolds which make constant angle with Y (see definition 3.1). We proved in Theorem 4.4 that if $L \subset S\partial(M, Y)$ is a totally geodesic submanifold of M , then it is a helix submanifold of N with respect to Y . A special case of helix submanifold is when the constant angle is orthogonal. In theorem 4.3 we proved that if $L \subset S\partial(M, Y)$ is orthogonal helix of N , then L is a totally geodesic submanifold of M .

Theorem 3.1 might be of help to develop some intuition about helix submanifolds of codimension one. Theorem 2.2 implies that a submanifold of N which is helix with respect to $\text{cod}-M$ parallel vector fields is a totally geodesic submanifold of N . Also Corollary 3.2 proves that a closed orientable helix surface in a three-dimensional manifold should be a torus or a totally geodesic submanifold.

Minimal submanifolds are relatives of totally geodesic submanifolds. A submanifold contained in a shadow boundary, $L \subset S\partial(M, Y)$, might be a minimal submanifold of M : Theorem 5.1 tell us that a necessary and sufficient condition is that the mean curvature vector field of $L \subset N$ should be orthogonal to Y . Finally, if $N = \mathbb{R}^{n+1}$ and $L \subset S\partial(M, Y)$ is compact, minimal of codimension one in M and L is contained in a totally geodesic submanifold of N , then L is totally geodesic in M .

Notation 1.1 We will work on the C^∞ category. In this manuscript (N, g) will be a Riemannian manifold with covariant derivative ∇ . We will use $\mathfrak{X}(N)$ to denote vector fields on N .

2 Parallel vector fields along submanifolds

Let $M \subset N$ be a submanifold. For every $x \in M$, there is a decomposition:

$$T_x N = T_x M \oplus T_x M^\perp.$$

This is a direct sum, i.e., there is a unique decomposition for every $V \in T_x N$: $V = \tan(V) + \text{nor}(V)$. Where $\tan(V) \in T_x M$ and $\text{nor}(V) \in T_x M^\perp$. From this, we can define two natural applications, $\tan : TN \longrightarrow TM$ and $\text{nor} : TN \longrightarrow TM^\perp$.

So, every vector field $Y : M \longrightarrow TN$ along M may be decomposed as $Y = \tan(Y) + \text{nor}(Y)$ into two vector fields (see [13] for details).

Notation 2.1 We will use $\mathfrak{X}(N)$ to denote the vector fields on N . Given a submanifold M of N , we can consider vector fields in N along M :

$\mathfrak{X}(N, M) = \{Y : M \longrightarrow TN\}$. Each $Y \in \mathfrak{X}(N, M)$ induces two natural vector fields $\tan(Y) : M \longrightarrow TM$ and $\text{nor}(Y) : M \longrightarrow TM^\perp$.

Let us remember the classic definition of a parallel vector field on a manifold.

Definition 2.1 A vector field $X \in \mathfrak{X}(N)$ is *parallel* if $\nabla_W X = 0$ for every $W \in TN$.

Example 2.1 Let $N = \mathbb{R} \times M$ be a Riemannian product manifold. Then N admits a parallel vector field defined by $Y : N \longrightarrow TN$ with $Y(t, y) = \partial_t$, where $T_{(t,y)}N = T_t\mathbb{R} \oplus T_yM$.

Remark 2.1 In [17] and [18], D. J. Welsh studied the existence of a parallel vector field on a Riemannian manifold. In [17] he gives the following criterion.

A complete and connected Riemannian manifold N admits p linearly independent parallel vector fields if and only if there exists a Riemannian manifold M_2 , and a group $L \subset \mathbb{R}^p \times I(M_2)$ such that

- (a) the first projection $pr|L$ is injective and
- (b) the orbits of L in $\mathbb{R}^p \times M_2$ are discrete, so N is isometric to $(\mathbb{R}^p \times M_2)/L$.

Definition 2.2 Let M be a Riemannian submanifold of N and let $Y \in \mathfrak{X}(N, M)$. We will say that Y is a *parallel vector field along M* , if $\nabla_W Y = 0$ for every $W \in TM$.

Notation 2.2 We will denote by $\mathfrak{X}_0(N, M)$ the set of all vector fields $Y : M \longrightarrow TN$ which are parallel along M .

Example 2.2 Let us assume that N admits a (global) parallel vector field $Y \in \mathfrak{X}(N)$. If M is a submanifold of M then $Y|_M : M \longrightarrow TN$ is a parallel vector field along M . Let us consider another example. Let N be a submanifold of \mathbb{R}^n and let H be any linear submanifold of \mathbb{R}^n tangent to N . Assume that $M = N \cap H$ is a submanifold. Then any constant vector field X in H induces a parallel vector field along M , $Y = X|_M : M \longrightarrow TN$, where Y it is not necessarily the restriction of a parallel vector field on N .

Remark 2.2 Let $M \subset N$ be a submanifold of codimension one. Let assume that $Y : M \longrightarrow TN$ is a parallel vector field along M . If Y tangent to M (i.e. $Y \in \mathfrak{X}(M)$), then Y is a parallel vector field on M . If Y is orthogonal to M , the it is parallel with respect to the normal connection ∇^\perp on TM^\perp . Let us observe that the converse assertions are false. But if M is a totally geodesic submanifold, then a parallel vector field on M (connection on M) or normal to M (normal connection) is parallel along M (connection on N).

Remark 2.3 Let $M \subset N$ be a Riemannian submanifold with induced connection ∇^M . Let $II_x : T_xM \times T_xM \longrightarrow T_xM^\perp$ be the second fundamental form of $M \subset N$ at $x \in M$. Finally, let ∇^\perp the induced normal connection on TM^\perp .

If $X : M \longrightarrow TM$ is a parallel vector field on M ($\nabla^M X = 0$) and for every $x \in M$, $II_x(X(x), \cdot) = 0$, then X is parallel along M , i.e. $\nabla_W X = 0$ ($W \in T_x M$).
If $Z : M \longrightarrow TM^\perp$ satisfies $\nabla^\perp Z = 0$ and for every $x \in M$, $g(Z(x), II_x(\cdot, \cdot)) = 0$, then Z is parallel along M , i.e. $\nabla_W Z = 0$ ($W \in T_x M$).

Remark 2.4 Let M be a connected submanifold of N . If Y is a parallel vector field along M , then $g(Y, Y)$ is constant: Let α be a smooth curve in M through $p \in M$. So, $\frac{d}{dt}g(Y(\alpha(t)), Y(\alpha(t))) = 2g(Y(\alpha), \nabla_{\dot{\alpha}} Y(\alpha)) = 0$ because Y is parallel along M . Therefore $g(Y, Y) = g(Y(p), Y(p))$ and since M is connected, there exist a smooth curve in M from any point to p .

Definition 2.3 Let M be a Riemannian submanifold of N . For every $y \in N$, we denote by $Hol_y(N)$ the Holonomy group based at y of the Levi-Civita connection of (N, g) . This is a subgroup of $O(T_y N)$, i.e. its elements are isometries of $T_y N$. To describe $Hol_y(N)$, we should consider a loop γ based at y (piece-wise smooth closed path through y) and take the parallel transport P_γ along it. Then

$$Hol_y(N) = \{P_\gamma : T_y N \longrightarrow T_y N \mid \gamma \subset N \text{ loop based at } y\}.$$

Let us consider the following subgroup of $Hol_x(N)$, where $x \in M$,

$$Hol_x(N, M) = \{P_\gamma \in Hol_x(N) \mid \gamma \subset M\}.$$

We will call $Hol_x(N, M)$, the *Holonomy subgroup at x with respect to M* .
Let us remember that $Hol_y(N)$ acts in $T_y N$.

Theorem 2.1 Let $M \subset N$ be a submanifold and let $G = Hol_x(N, M)$, where $x \in M$. There exist a parallel vector field along M , $X : M \longrightarrow TN$, if and only if there exist $W \in T_x N$ such that $G_W = G$, where G_W is the isotropy subgroup at x under the action of G in $T_x N$.

Proof. Let $W \in T_x N$ be a fixed vector under the action of G in $T_x N$. For every $z \in M$, we define $X(z) = P_\beta(W)$, where $\beta : [0, 1] \longrightarrow M$ is any piece-wise smooth regular curve with $x = \beta(0)$ and $z = \beta(1)$. Affirmation: X does not depends on β . Let α be another curve with the same conditions as β . Let us consider the next loop

$$\gamma(t) = \begin{cases} \tilde{\beta}(t) = \beta(2t) & \text{if } t \in [0, \frac{1}{2}] \\ \tilde{\alpha}(t) = \alpha(2 - 2t) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

By hypothesis, $P_\gamma(W) = W$, i.e., $P_{\tilde{\alpha}}(P_{\tilde{\beta}}(W)) = W$. It follows that $P_{\tilde{\beta}}(W) = P_{\tilde{\alpha}}^{-1}(W)$. Then $P_\beta(W) = P_\alpha(W)$. Here we used that we have a homomorphism group from $\pi_1(N, x)$ into $Hol_x(N)/Hol_0(N)$, where $Hol_0(N)$ is the restricted Holonomy group (null-homotopic loops, see [3], page 280).

The smoothness of X follows from observations in Besse's book [3], page 282. \square

Definition 2.4 Let $M \subset N$ be a Riemannian submanifold. If every geodesic of M is a geodesic of N , then M is called a *totally geodesic submanifold (tgs)*.

Theorem 2.2 *Let $M \subset N$ be a Riemannian submanifold of codimension r . Let $X_j : M \longrightarrow TN$, $j = 1, \dots, r$, be parallel vector fields along M , such that for every $x \in M$, $\{X_1(x), \dots, X_r(x)\}$ is a basis of $T_x M^\perp$. Then M is a totally geodesic submanifold of N .*

Proof. Let $x \in M$, by hypothesis, $\dim(T_x M^\perp) = r$, and

$$T_x N = T_x M \oplus (\oplus_{j=1}^r \langle X_j(x) \rangle). \quad (1)$$

Let $\gamma \subset M$ be a geodesic through x , so $\nabla_{\dot{\gamma}} \dot{\gamma}$ is orthogonal to M , i.e., $\nabla_{\dot{\gamma}} \dot{\gamma} \in T_{\gamma} M^\perp$. Affirmation: γ is geodesic of N . By equality (1), we need prove that $\nabla_{\dot{\gamma}} \dot{\gamma}$ is orthogonal to every $X_j(\gamma)$: Since $g(X_j(\gamma), \dot{\gamma}) = 0$,

$$g(X_j(\gamma), \nabla_{\dot{\gamma}} \dot{\gamma}) + g(\nabla_{\dot{\gamma}} X_j(\gamma), \dot{\gamma}) = dg(X_j(\gamma), \dot{\gamma}) = 0.$$

Therefore, $\nabla_{\dot{\gamma}} \dot{\gamma}$ is orthogonal to N . Hence, γ is a geodesic of N . \square

3 Helix submanifolds

The next definition is a natural extension of the classic concept of general helix in \mathbb{R}^3 which appears from the first courses in differential geometry: a curve in \mathbb{R}^3 which makes constant angle with respect to a fixed direction. There are extensions into a three manifold, but the helix is again a curve (see [1] and [5]). In the following definition a helix may be a submanifold of higher dimension.

Definition 3.1 Let M be a Riemannian submanifold of N and let $Y \in \mathfrak{X}_0(N, M)$ be a parallel vector field along M . We say that M is a *helix submanifold* of N with respect to Y if the following function $h : M \longrightarrow \mathbb{R}$ is constant.

$$h(x) = \max\{g(w, Y(x)) | w \in T_x M, g(w, w) = 1\}. \quad (2)$$

Remark 3.1 We could think of $h(x)$ as the angle between $T_x M$ and $Y(x)$. Let us observe that $h(x) = g(\frac{\tan(Y(x))}{(g(\tan(Y(x)), \tan(Y(x))))^{1/2}}, Y(x)) = (g(\tan(Y(x)), \tan(Y(x))))^{1/2}$.

Example 3.1 Let $M \subset N$ be a connected and totally geodesic submanifold. If $Y \in \mathfrak{X}_0(N, M)$ then M is a helix submanifold of N with respect to Y . By Theorem 2.1, Y is invariant under parallel transport on N along curves contained on M . Is well known that TM is also invariant under parallel transport on N . So the angle between Y and TM is constant.

Example 3.2 Let $\gamma \subset N$ be a embedded geodesic. Since the tangent vector $\dot{\gamma}$ of γ is parallel along it, γ is a helix of N with respect to any parallel vector field ($\dot{\gamma}$ itself) along γ .

A helix submanifold is not necessarily a totally geodesic submanifold, for example a circular cylinder and any cone of revolution in \mathbb{R}^3 are helix submanifolds with respect to a constant vector field parallel to their axis.

Lemma 3.1 *Let M be a connected helix submanifold of N with respect to $Y \in \mathfrak{X}_0(N, M)$. Then $\tan(Y)$ and $\text{nor}(Y)$ have constant length.*

Proof. Since M is connected, Y has constant length.

By hypothesis, the function h in (2) is constant, so remark 3.1 implies that $h(x) = (g(Y_0(x), Y_0(x)))^{1/2}$ is constant, where $Y_0(x) = \tan(Y(x))$. To prove that $\text{nor}(Y)$ has constant length, we can use the decomposition $Y = \tan(Y) + \text{nor}(Y)$ and the equality $g(Y, Y) = g(\tan(Y), \tan(Y)) + g(\text{nor}(Y), \text{nor}(Y))$. \square

Corollary 3.1 *Let M be a compact, orientable and connected submanifold of N . If M is a helix submanifold with respect to Y , then it has zero Euler characteristic or Y is orthogonal to M .*

Proof. Let us assume that Y is not orthogonal to M . Then by Lemma 3.1, $\tan(Y) \in \mathfrak{X}(M)$ is a vector field on M of constant length. Since M is compact and orientable we conclude, by a well known Poincare-Hopf's theorem (see [7]), that M has zero Euler characteristic. \square

Corollary 3.2 *Let M be a connected, orientable and compact submanifold of dimension two of N^3 . If M is a helix of N , then M is diffeomorphic to a torus or it is totally geodesic.*

Proof. Let assume that M is a helix submanifold with respect to $Y \in \mathfrak{X}_0(N, M)$, a parallel vector field along M . By Corollary 3.1, M has zero Euler characteristic or Y is orthogonal to Y . In the first case, we can deduce that M is a torus. In the second case, M has an orthogonal vector field. Since Y is parallel along M , then M is totally geodesic. \square

Example 3.3 Let us consider a connected hypersurface M in $N = \mathbb{R}^{n+1}$. Let $Y \in \mathfrak{X}(N)$ be a constant vector field. If M is a helix submanifold of N , then

- M is contained in a hyperplane (perpendicular to Y) of N when Y is orthogonal to M .
- M is not compact (otherwise Y would be orthogonal to M),
- M is orientable ($\text{nor}(Y)$ induces an orientation),
- M has zero Gauss-Kronecker curvature (the Gauss map of M is singular).

If $M \subset \mathbb{R}^{n+1}$ is not a hypersurface but is compact, we can conclude that M is contained in a hyperplane orthogonal to Y .

Proposition 3.1 *If M is a compact helix of $N = \mathbb{R} \times M_2$ with respect to $X = \partial_t$ then X is orthogonal to M .*

Proof. Since M is compact, the projection π_1 of M in \mathbb{R} is compact, so $\pi_1(M) \subset \mathbb{R}$ has a maximum denoted by t_0 . Let $x \in M$ be such that $\pi_1(x) = t_0$, affirmation: $t_0 \times M_2 = \pi_1^{-1}(t_0)$ is tangent to M in x . We deduce from this that $T_x M \subset T_x(t_0 \times M_2)$. Let us observe that X is orthogonal to $t_0 \times M_2$, in consequence X is orthogonal to M at x . Since M is a helix, X is orthogonal to M . \square

Remark 3.2 In general a compact helix submanifold M , with respect to a global parallel vector field X on N , is not necessarily orthogonal to X . Using Example 3.1, we can construct an example of this: M should be a totally geodesic submanifold of N and X a global parallel vector field on N but non-orthogonal, for example tangent along M .

For general Riemannian hypersurfaces which are helix, we can prove the following result.

Theorem 3.1 *Let M be a connected hypersurface in a Riemannian manifold N . Let us assume that M is a helix submanifold of N with respect to $Y \in \mathfrak{X}_0(N, M)$. Then*

a) If Y is orthogonal to M at some point, then

M is totally geodesic submanifold of N .

b) If Y is tangent to M at some point, then

M is locally a Riemannian product $\mathbb{R} \times M_2$.

c) If Y is transversal to M at some point and the integral curves of $\tan(Y)$ are geodesics in M , then they are geodesics in N .

Proof. Since M is a helix, the angle between Y and M is constant.

a). We have that Y is parallel along M and orthogonal to M . So, M is a totally geodesic submanifold of N .

b). Let us observe that Y is a parallel vector field on M , then by Welsh's work in [17], M is locally isometric to a Riemannian product.

c). In this case, Y is transversal to M in any point. Let $Y_0 = \tan(Y)$ and let $\alpha \subset M$ be an integral curve of Y_0 , i.e. $\dot{\alpha}(t) = Y_0(\alpha(t))$. Affirmation: Y is orthogonal to $\nabla_{\dot{\alpha}}\dot{\alpha}$.

$$0 = \frac{d}{dt}g(\dot{\alpha}, Y_0(\alpha)) = \frac{d}{dt}g(\dot{\alpha}, Y(\alpha)) = g(\nabla_{\dot{\alpha}}\dot{\alpha}, Y) + g(\dot{\alpha}, \nabla_{\dot{\alpha}}Y) = g(\nabla_{\dot{\alpha}}\dot{\alpha}, Y),$$

where $\nabla_{\dot{\alpha}}Y = 0$ because Y is parallel along M . Since α is geodesic in M , $\nabla_{\dot{\alpha}}\dot{\alpha}$ is orthogonal to M . So, $\nabla_{\dot{\alpha}}\dot{\alpha} = 0$, otherwise Y would be tangent to M but it is transversal to M . \square

Remark 3.3 Let $M \subset N$ be a submanifold of codimension one and let $Y \in \mathfrak{X}_0(N, M)$ be transverse (important) to M . If every geodesic of M is a helix of N with respect to Y , then M is a totally geodesic submanifold of N . Proof: Let γ be a geodesic on M ($\nabla_{\dot{\gamma}}\dot{\gamma} \in TM^\perp$), then the equation

$$g(Y(\gamma(t)), \nabla_{\dot{\gamma}}\dot{\gamma}) + g(\nabla_{\dot{\gamma}}Y(\gamma(t)), \dot{\gamma}) = \frac{d}{dt}g(Y(\gamma(t)), \dot{\gamma})$$

and the hypothesis imply that $g(Y(\gamma(t)), \nabla_{\dot{\gamma}}\dot{\gamma}) = 0$. Since M is of codimension one, $T_{\gamma(t)}N = \langle Y(\gamma(t)) \rangle \oplus T_{\gamma(t)}M$. Therefore, $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$, i.e. γ is a geodesic of N .

If Y is not transverse to M , we can not conclude that M is a totally geodesic submanifold of N . For example a circular cylinder M in $N = \mathbb{R}^3$. Their geodesics are helix (and M itself) with respect to a constant vector field in the direction of its axis.

Remark 3.4 Let $N = \mathbb{R}^3$ and let $v \in N$ be a non-zero vector. If $M^2 \subset N$ is a complete minimal surface which is a helix with respect to $Y = v$, then M is a plane. Proof: By Example 3.3, M has zero Gauss-Kronecker curvature, but in dimension 2 it is the Gaussian curvature. Since M is minimal, it easy to see that M is a plane.

Question 3.1 If $M^n \subset N^{n+1}$ is minimal and a helix submanifold, does it is a totally geodesic submanifold of N ?

Remark 3.5 The argument in Remark 3.4, when $N = \mathbb{R}^3$, is not valid in $N = \mathbb{R}^{n+1}$ with $n \geq 3$: In \mathbb{R}^4 there are minimal hypersurfaces with zero Gauss-Kronecker curvature, like $M^3 = M' \times \mathbb{R}$, where M' is minimal in \mathbb{R}^3 .

4 Shadow boundary and helix

Definition 4.1 Let M be a Riemannian submanifold of N , and let $Y : M \longrightarrow TN$ be a parallel vector field along M (i.e. $Y \in \mathfrak{X}_0(N, M)$). The *shadow boundary* of M with respect to Y is the following subset of M .

$$S\partial(M, Y) = \{x \in M \mid Y(x) \in T_x M\}. \quad (3)$$

Remark 4.1 The shadow boundary is a natural subset of M , it is the locus where Y is tangent to M .

The subset $S\partial(M, Y) \subset M$ is closed, so if M is compact it is also compact. $S\partial(M, Y)$ is not always a submanifold of M . $S\partial(M, Y)$ may be empty (when Y is nowhere tangent to M), or equal to M (when Y is anywhere tangent to M).

Example 4.1 Let $N = S^1 \times S^1 \times S^1$ be the three-torus. Consider the totally geodesic submanifold $M = S^1 \times S^1 \times \{x\} \subset N$, where $x \in S^1$. Let $Y : M \longrightarrow TM$ be a parallel vector field on M . Then Y is parallel along M . So, $S\partial(M, Y) = M$.

Now take $Y' : M \longrightarrow TM^\perp$ orthogonal to M . Again Y' is parallel along M and $S\partial(M, Y') = \emptyset$.

Finally, when $N = \mathbb{R}^n$ and M is a compact submanifold, the shadow boundary with respect to any constant global vector field Y on N is non-empty.

Theorem 4.1 Let M be the Riemannian product $M_1 \times M_2$, of two submanifolds $M_1 \subset N_1$ and $M_2 \subset N_2$. Let $Y = (Y_1, Y_2)$ where $Y_j \in \mathfrak{X}_0(N_j, M_j)$. Then

$$S\partial(M, Y) = S\partial(M_1, Y_1) \times S\partial(M_2, Y_2).$$

Proof. Let $x = (x_1, x_2) \in S\partial(M, v)$, it means, that $(Y_1, Y_2) \in T_x M \simeq T_{x_1} M \oplus T_{x_2} M$. Which is equivalent to $Y_1 \in T_{x_1} M$, $Y_2 \in T_{x_2} M$. Therefore, $x_1 \in S\partial(M_1, Y_1)$ and $x_2 \in S\partial(M_2, Y_2)$. This concludes the proof. \square

Example 4.2 Let $M = S^1 \times S^1 \subset S^2 \times S^2$. By Theorem 4.1, we have the following cases: Let $Y_j \in \mathfrak{X}_0(S^2, S^1)$ ($j=1, 2$) be parallel vector fields along S^1 .

Let $Y = (Y_1, 0)$, with $Y_1 \neq 0$ and $0 \in \mathfrak{X}_0(S^2, S^1)$. Let us observe that $S\partial(S^1, Y_1) = S^1$ or $\emptyset \subset S^2$ and $S\partial(S^1, 0) = S^1 \subset S^2$. Then $S\partial(M, Y) = S^1 \times S^1$ or \emptyset .

Similarly, if $Y = (0, Y_2)$, $S\partial(M, Y) = S^1 \times S^1$ or \emptyset . Finally $Y = (Y_1, Y_2)$, with $Y_1 \neq 0, Y_2 \neq 0$. Then $S\partial(M, Y) = S^1 \times S^1$ or \emptyset .

The second fundamental form of $M \subset N$ at $x \in M$ is a symmetric bilinear tensor, which we denote by $II_x : T_x M \times T_x M \longrightarrow T_x M^\perp$. So, II_x is a bilinear application for every $x \in M$ (see [11] page 12, for details).

Let Y be a parallel vector field along M . Let $x \in M$ such that $Y(x) \in T_x M$, then we can consider the following linear application:

$$II(Y(x), \cdot) : T_x M \longrightarrow T_x M^\perp.$$

If this transformation is surjective we will say that $II(Y(x), \cdot)$ is surjective. In particular, if $\text{cod} M = 1$, the latter condition is equivalent to $II(Y(x), \cdot) \neq 0$.

Theorem 4.2 *Let M be a submanifold of dimension n and codimension k in N ($n \geq k$). Let Y be a parallel vector field along M . If $II(Y(y), \cdot)$ is surjective for every $y \in S\partial(M, Y)$, then $S\partial(M, Y)$ is a submanifold of dimension $n - k$ in M .*

Proof.

Let ∇ be the covariant derivative of N . Let $p \in S\partial(M, Y)$, and let $U \subset M$ be a open neighborhood of p . Affirmation: $S\partial(M, Y) \cap U$ is a submanifold of M .

Let $\xi_j : U \longrightarrow TU^\perp$, $j = 1, \dots, k$ be a basis of orthonormal vector fields (U is such that there exist these vector fields). Let us consider the next function $F : U \longrightarrow \mathbb{R}^k$, given by

$$F(x) = (g(Y(x), \xi_1(x)), \dots, g(Y(x), \xi_k(x))).$$

Is clear that $F^{-1}(0) = S\partial(M, Y) \cap U$. We are going to prove that $0 \in \mathbb{R}^k$ is a regular value of F . We need verify that for every $x \in S\partial(M, Y) \cap U$, $F_{*x} : T_x M \longrightarrow \mathbb{R}^k$ is surjective. Let (y_1, \dots, y_n) be local coordinates in U . Let us calculate the next derivatives in these coordinates, $\frac{\partial F}{\partial y_l} = (\frac{\partial}{\partial y_l} g(Y(x), \xi_1(x)), \dots, \frac{\partial}{\partial y_l} g(Y(x), \xi_k(x)))$, for every $1 \leq l \leq n$. Since, Y is parallel,

$$\frac{\partial}{\partial y_l} g(Y(x), \xi_j(x)) = g(\nabla_{\partial y_l} Y, \xi_j) + g(Y, \nabla_{\partial y_l} \xi_j) = g(Y, \nabla_{\partial y_l} \xi_j).$$

Let us apply Weingarten's formula, which says that

$\nabla_{\partial y_l} \xi_j = -A_{\xi_j}(\partial y_l) + \nabla_{\partial y_l}^\perp \xi_j$. In conclusion,

$$\frac{\partial}{\partial y_l} g(Y(x), \xi_j(x)) = g(Y, -A_{\xi_j}(\partial y_l)) = -g(II(Y, \partial y_l), \xi_j(x)),$$

for every $x \in S\partial(M, Y)$, $1 \leq j \leq k$, and $1 \leq l \leq n$.

Now we are ready to see that the next matrix

$$(F_{*x})_{jl} = -(g(II(Y, \partial y_l), \xi_j(x)))$$

has rank k . Let us assume that the row vectors are linearly dependent, i.e. we have the following condition

$\sum_{j=1}^k a_j g(II(Y, \partial y_l), \xi_j(x)) = 0$, for every $1 \leq l \leq n$, and where $a_j \in \mathbb{R}$ are constants. We can rewrite this expression as

$$g(II(Y, \partial y_l), \sum_{j=1}^k a_j \xi_j(x)) = 0,$$

for every $1 \leq l \leq n$. Since $II(Y, \cdot)$ is surjective, $\sum_{j=1}^k a_j \xi_j(x) = 0$, therefore $a_j = 0$. Which proves that 0 regular value of F . Then we can conclude that, $F^{-1}(0) \cap U$ is a submanifold U of dimension $n - k$. \square

Remark 4.2 A special case of Theorem 4.2 is when $\dim N = 2 \dim M$. The conclusion in this situation is that $S\partial(M, Y)$ is a discrete subset of M . So if M were compact, $S\partial(M, Y)$ would be a finite set of points in M .

Remark 4.3 In [15], we proved that when $M^n \subset \mathbb{R}^{n+1}$ has nowhere zero Gauss-Kronecker curvature, then for every $v \in \mathbb{R}^{n+1}$, $S\partial(M, v)$ is a submanifold of M of codimension one.

In the particular case of surfaces in \mathbb{R}^3 , the smoothness of some shadow boundaries is studied in [6], [10] and [9].

Definition 4.2 Let $L \subset N$ be a Riemannian submanifold. Let $x \in L$, then L is called a *totally geodesic submanifold of N at the point x* if every geodesic γ of L through x satisfies $\nabla_{\dot{\gamma}} \dot{\gamma}|_x = 0$.

In her work on Affine Differential Geometry [16], A. Schwenk used conditions similar to those of the next Theorem.

Theorem 4.3 Let $M^n \subset N^{n+k}$ ($n \geq 2$) be a submanifold of codimension k ($k \geq 0$). Let $L \subset M$ be a submanifold of codimension one, which is not totally geodesic of N at any point. Let Y be a parallel vector field along M and orthogonal to L . Then $L \subset S\partial(M, Y)$ if and only if L is a totally geodesic submanifold of M .

Proof.

\implies) Let $x \in L$, since $\dim(T_x L^\perp \cap T_x M) = 1$ and by hypothesis $Y(x) \in T_x L^\perp \cap T_x M$, we obtain that $\langle Y(x) \rangle = T_x L^\perp \cap T_x M$. Therefore, we have the following equality for every $x \in L$,

$$T_x M = T_x L \oplus (T_x L^\perp \cap T_x M) = T_x L \oplus \langle Y(x) \rangle. \quad (4)$$

Let $\gamma \subset L$ be a geodesic and let $x \in \gamma$ be any point. Hence $\nabla_{\dot{\gamma}} \dot{\gamma}$ is orthogonal to L , i.e. $\nabla_{\dot{\gamma}} \dot{\gamma} \in T_x L^\perp$. Affirmation: γ is a geodesic of M . By equality (4), we just have to verify that $\nabla_{\dot{\gamma}} \dot{\gamma}$ is orthogonal to $Y(x)$: We know that $g(Y(\gamma(t)), \dot{\gamma}) = 0$, this implies that,

$$g(Y(\gamma(t)), \nabla_{\dot{\gamma}} \dot{\gamma}) + g(\nabla_{\dot{\gamma}} Y(\gamma(t)), \dot{\gamma}) = \frac{d}{dt} g(Y(\gamma(t)), \dot{\gamma}) = 0.$$

Then, $\nabla_{\dot{\gamma}}\dot{\gamma}$ is orthogonal to M , so γ is a geodesic of M .

\Leftarrow) In this implication we assume that $k = 1$. Let $x \in L$, affirmation: $Y(x) \in T_x M$. Since L is not a totally geodesic submanifold of N at x , there exists a geodesic γ of L through x with $\nabla_{\dot{\gamma}}\dot{\gamma}|_x \neq 0$. By hypothesis, γ is also a geodesic of M . So, $\nabla_{\dot{\gamma}}\dot{\gamma} \in (T_{\gamma} M)^\perp$. Let us prove that $Y(x)$ is orthogonal to $\nabla_{\dot{\gamma}}\dot{\gamma}$. For this, let us observe that $g(\dot{\gamma}, Y(x)) = 0$. Therefore,

$$g(Y(\gamma(t)), \nabla_{\dot{\gamma}}\dot{\gamma}) + g(\nabla_{\dot{\gamma}}Y(\gamma(t)), \dot{\gamma}) = \frac{d}{dt}g(Y(\gamma(t)), \dot{\gamma}) = 0.$$

But $\nabla_{\dot{\gamma}}Y(\gamma(t)) = 0$, because Y is parallel along L . Since M is of codimension one, $g(Y(\gamma(t)), \nabla_{\dot{\gamma}}\dot{\gamma}) = 0$ implies that $Y(x) \in T_x M$. \square

Remark 4.4 In Theorem 4.3, the condition that L is not totally geodesic in N at any point is important to prove that $L \subset S\partial(M, Y)$. We can see this with the next example: $N = \mathbb{R}^n$, M a hyperplane, L a linear subspace of codimension one in M . Finally let $Y = v$ be any constant vector field orthogonal to M . In this example the affirmation $L \subset S\partial(M, Y)$ is false.

The next Theorem was the original motivation to consider Helix submanifolds in this work.

Theorem 4.4 *Let $M^n \subset N^{n+k}$ ($n \geq 2$) be a submanifold of codimension k ($k \geq 0$). Let $L \subset M$ be a submanifold and let $Y \in \mathfrak{X}_0(N, M)$. Assume that $L \subset S\partial(M, Y)$. If L is a totally geodesic submanifold of M , then L is a helix submanifold of N with respect to Y .*

Proof.

If $Y(x) \in T_x L$, for every $x \in L$, L is a helix. Otherwise, let $p \in L$ such that $Y(p) \notin T_p L$. So,

$$T_p L \oplus \langle Y(p) \rangle \subset T_p M. \quad (5)$$

Affirmation: The angle between $T_x L$ and $Y(x)$ is constant, for every x in L . Let γ be any geodesic of L from p to x , hence it is also geodesic of M . Now, let us consider the parallel transport τ in M , along γ , from p to x . Therefore, $\tau : T_p M \longrightarrow T_x M$ is an isometry. So, τ transforms the latter equation (5), in $T_x L \oplus \langle Y(x) \rangle \subset T_x M$. Since the parallel transport is an isometry, the angle between $T_x L$ and $Y(x)$ is equal to the angle between $T_p L$ and $Y(p)$. \square

Remark 4.5 Let us observe that any Euclidean compact helix with respect to $Y = v \in \mathbb{R}^{n+k} - \{0\}$ should be orthogonal to Y . So, if $N = \mathbb{R}^{n+k}$ and $L \subset S\partial(M, Y) \subset M$ is a compact totally geodesic submanifold of M , then by Theorem 4.4, L is orthogonal to Y .

5 Minimal shadow boundaries

Definition 5.1 Let $L \subset M$ be a Riemannian submanifold. Let $x \in L$, $n = \dim L$ and let e_1, \dots, e_n be an orthonormal basis of $T_x L$. The *mean curvature vector field* H of $L \subset M$ at x , is

$$H(x) = \frac{1}{n} \sum_{i=1}^n II_x(e_i, e_i).$$

Remark 5.1 For every $x \in L$, $H(x) \in T_x L^\perp$, because $II_x(e_i, e_i) \in T_x L^\perp$ where $i = 1, \dots, n$.

Definition 5.2 A submanifold $L \subset M$ is *minimal* if $H(x) = 0$ for every $x \in L$.

The next proposition and its proof is due to C. Bang-yen, see [2].

Proposition 5.1 (*Bang's Lemma*) Let L^n be a submanifold of M^s , where M is a submanifold of N^m . Then L is minimal in M if and only if the mean curvature vector field of $L \subset N$ is orthogonal to M .

Proof. Let X and Y be two vector fields on L . Let ∇ and ∇' be the covariant derivatives of N and M respectively. Gauss formula for $M \subset N$ says that

$$\nabla_X Y = \nabla'_X Y + II^N(X, Y),$$

where II^N is the second fundamental form of $M \subset N$.

Let ∇'' be the covariant derivative of L and II^M the second fundamental form of $L \subset M$. Then we have

$$\nabla'_X Y = \nabla''_X Y + II^M(X, Y).$$

From the two latter formulae we get

$$\nabla_X Y = \nabla''_X Y + II^M(X, Y) + II^N(X, Y).$$

So, the second fundamental form II of $L \subset N$ is

$$II(X, Y) = II^M(X, Y) + II^N(X, Y).$$

By definition, II^M is orthogonal to L , tangent to M and II^N is orthogonal to M . Now, let us consider the mean curvature vector fields. Let H and H^M be the mean curvature vector fields of $L \subset N$ and $L \subset M$ respectively. Let $x \in L$ and let e_1, \dots, e_n be an orthonormal of $T_x L$. Then $\frac{1}{n} \sum_{i=1}^n II(e_i, e_i) = \frac{1}{n} \sum_{i=1}^n II^M(e_i, e_i) + \frac{1}{n} \sum_{i=1}^n II^N(e_i, e_i)$. Hence,

$$H(x) = H^M(x) + H(L; M, N),$$

where $H(L; M, N) = \frac{1}{n} \sum_{i=1}^n II^N(e_i, e_i)$. Let us observe that $H(L; M, N)$ is orthogonal to M . Then L is minimal in M if and only if $H(x) = H(L; M, N)(x)$. \square

Theorem 5.1 *Let $M^n \subset N^{n+1}$ be a Riemannian submanifold and let $L^{n-1} \subset M$ be a submanifold such that $L \subset S\partial(M, Y)$, where Y is parallel along M and transverse to L . Let H be the mean curvature vector field of $L \subset N$. Then L is minimal in M if and only if $g(H, Y) = 0$.*

Proof. By hypothesis $Y(x) \in T_x M$ for every $x \in L$. By Lemma 5.1, if L is minimal in M then H is orthogonal to M . So, $H(x)$ is orthogonal to $Y(x)$, i.e. $g(H, Y) = 0$. Now, let us assume that $g(H, Y) = 0$. By definition, H is orthogonal to L . To apply Proposition 5.1, we need prove that H is orthogonal to M . Since Y is transversal to L , $T_x M = T_x L \oplus \langle Y(x) \rangle$ for every $x \in L$. Now it is clear that H is orthogonal to M . Then L is minimal in M . \square

Corollary 5.1 *Let $N = \mathbb{R}^{n+1}$, let $M^n \subset N$ be a submanifold and let Y be a constant vector field on N induced by $v \in \mathbb{R}^{n+1} - \{0\}$. Let $L \subset S\partial(M, Y)$ be a compact submanifold of M of codimension one. Let us assume that*

- *L is contained in a hyperplane, which is transverse to v .*

If L is minimal then L is a totally geodesic submanifold of M .

Proof. Let Π be the hyperplane that contains L . Hence the mean curvature vector H , of $L \subset \mathbb{R}^{n+1}$, is contained in Π : $H_p \in T_p \Pi$ for every $p \in L$. Since Π is transversal to v , then v is not tangent to L . Now we can apply Theorem 5.1 (M is minimal), to deduce that $\langle H(p), v \rangle = 0$, for every $p \in L$. In summary, v is orthogonal to $H(p)$, where $H(p) \in T_p \Pi$.

Affirmation: v is orthogonal to Π . Since L is compact, we can get a basis of $T_p \Pi$, by translation of vectors $H(x)$ into the point p . Moreover, $H(x)$ can not be zero for every x because L is compact and N do not contains minimal compact submanifolds. We proved that $H(x)$ is orthogonal to v , for every x in L . This proves that v is orthogonal to Π . Since $L \subset \Pi$, v is orthogonal to L .

Finally, we can apply Theorem 4.3, which says that if $L \subset S\partial(M, Y)$ and v is orthogonal to L , then L is a totally geodesic submanifold of M . \square

Example 5.1 In this example we are going to construct a submanifold M of $N = \mathbb{R}^{n+2}$, which contains a minimal submanifold in some shadow boundary.

Let $L^n \subset \mathbb{R}^{n+2}$ be a submanifold and let $Y = v \in \mathbb{R}^{n+2} - \{0\}$ such that:

- v is transverse to L : $v \notin T_x L$ for every $x \in L$,
- $\langle H, v \rangle = 0$, where H is the mean curvature vector field of $L \subset N$.
- $L_{\epsilon, v} = \{y = x + \lambda v \in \mathbb{R}^{n+2} \mid x \in L, |\lambda| < \epsilon\}$ is a submanifold, where ϵ denotes a positive smooth function of L .

Then L is a minimal submanifold of $M = L_{\epsilon, v}$. If L is compact, ϵ can be a constant function. Proof: this is consequence of Theorem 5.1. We should verify the hypothesis of such theorem. The submanifold $M = L_{\epsilon, v}$ is a “cylindric” neighborhood of L in direction v . For this reason, $S\partial(M, v) = L_{\epsilon, v} = M$, and then $L \subset S\partial(M, v)$.

Example 5.2 Construction in Example 5.1 is based in a submanifold of codimension two, which is transverse to v and whose mean curvature vector field is orthogonal to v . There are submanifolds of codimension two which does not admits a transverse direction like $S^1 \times S^1 \subset \mathbb{R}^4$.

There are examples which satisfy conditions of Example 5.1:

1. Let $N_1 \subset N = \mathbb{R}^3$ be a classic helix with respect to direction v , in fact it can be a planar curve orthogonal to v . Let us assume that its curvature is nowhere zero. Since $N_1 \subset N$ is a helix, its mean curvature vector field, H_1 , is orthogonal to v . In fact $\|H_1\| \neq 0$ is its curvature.

Let us consider the surface $M = N_1 \times \mathbb{R} \subset \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$.

Affirmation: $\langle H_M, v \rangle = 0$ and $v \in \mathbb{R}^3 \times \{0\}$ is transverse to M , where H_M is the mean curvature vector of $M \subset \mathbb{R}^4$:

$H_M(x, y) = \frac{1}{2}H_1(x)$. Since v is transverse to $\{0\} \times \mathbb{R}$ and to N_1 , it is transverse to M .

2. It is not necessary to take a classic helix in the latter example. Here is another construction. Let $N_1 \subset \mathbb{R}^2$ be any embedded curve whose curvature is nowhere zero. Let $N_2 \subset \mathbb{R}^3$ be any minimal surface of \mathbb{R}^3 which admits a transverse direction $w \in \mathbb{R}^3$. Let $M^3 = N_1 \times N_2 \subset \mathbb{R}^2 \times \mathbb{R}^3 = \mathbb{R}^5$. Affirmation: $w \in \{0\} \times \mathbb{R}^3$ is transverse to M and $\langle H_M, w \rangle = 0$.

As before, $H_M(x, y) = \frac{1}{3}H_1(x) + \frac{2}{3}H_2(y) = \frac{1}{3}H_1(x)$. Now let us observe that $H_1(x) \in \mathbb{R}^2 \times \{0\}$, w is orthogonal to N_1 and transverse to N_2 .

Question 5.1 Does exist a compact strictly convex hypersurface in \mathbb{R}^{n+1} with some minimal and non-totally geodesic shadow boundary?

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